

# On the Optimality of Functionals over Triangulations of Delaunay Sets

Nikolay P. Dolbilin, Herbert Edelsbrunner and Oleg R. Musin\*

In this short paper, we consider the functional density on sets of uniformly bounded triangulations with fixed sets of vertices. We prove that if a functional attains its minimum on the Delaunay triangulation, for every finite set in the plane, then for infinite sets the density of this functional attains its minimum also on the Delaunay triangulations.

A *Delaunay set* in  $\mathbb{R}^d$  is a set of points,  $X$ , for which there are positive  $r$  and  $R$  such that every open  $d$ -ball of radius  $r$  contains at most one point and every closed  $d$ -ball of radius  $R$  contains at least one point of  $X$ . In this paper, we consider Delaunay sets in general position, i.e. no  $d + 2$  points in  $X$  lie on a common  $(d - 1)$ -sphere. By a *triangulation* of  $X$  we mean a simplicial complex whose vertex set is  $X$ . For finite sets, the simplices decompose the convex hull of the set, while for Delaunay sets  $X$ , the simplices decompose  $\mathbb{R}^d$ . We say a triangulation  $T$  is *uniformly bounded* if there exists a positive number  $q = q(T)$  that is larger than or equal to the circumradii of all  $d$ -simplices in the triangulation:  $\mathcal{R}(S) \leq q$  for all  $d$ -simplices  $S$  of  $T$ . We denote the family of all uniformly bounded triangulations of  $X$  by  $\Theta(X)$ .

Delaunay sets have been introduced by Boris Delaunay in 1924, who called them  $(r, R)$ -systems. He proved that for any Delaunay set  $X$ , there exists a unique Delaunay tessellation  $DT(X)$ ; see e.g. [1]. If  $X$  is in general position, then  $DT(X)$  is a triangulation of  $X$  in the sense defined above. Since the circumradius of any simplex is at most  $R$ , the Delaunay triangulation is uniformly bounded with  $q = R$ , i.e.  $DT(X) \in \Theta(X)$ . We note that every Delaunay set also has triangulations that are not uniformly bounded, and it is not difficult to construct them.

We want to remind the reader of a related open problem about Delaunay sets: *is it true that for every planar Delaunay set,  $X$ , and for every positive number,  $C$ , there exists a triangle  $\Delta$  that contains none of the points in  $X$  and whose area is greater than  $C$ ?* While we heard of this question from Michael Boshernitzan, it is sometimes referred to as *Danzer's problem*.

Let  $F$  be a functional defined on  $d$ -simplices  $S$ . For instance,  $F(S)$  may be the sum of squares of edge lengths multiplied by the volume of  $S$ . We only consider functionals that are continuous in parameters describing the simplices, for example the lengths of their edges.

---

\*This research is supported by the Russian government project 11.G34.31.0053, RFBR grant 11-01-00735, and DMS 1101688.

Let  $Y$  be a finite set in  $\mathbb{R}^d$  and  $T$  any triangulation of  $Y$ . Then  $F$  can be defined on  $T$  as

$$F(T) = \sum_{S \in T} F(S).$$

It is clear, that this definition cannot be used for infinite sets. We therefore define the (*lower*) *density* of  $F$  for a uniformly bounded triangulation  $T$  of a Delaunay set  $X$  as

$$\bar{F}(T) = \liminf_{\alpha \rightarrow \infty} \frac{1}{\text{vol}(B_\alpha)} \sum_{S \subset B_\alpha} F(S),$$

where  $B_\alpha$  denotes the closed ball of radius  $\alpha$  with center at the origin of  $\mathbb{R}^d$ . We call  $\bar{F}(T)$  the (*lower*) *density* of  $F$  on  $T$ . For the rest of the paper, we limit ourselves to dimension  $d = 2$ .

**Theorem.** *Let  $F$  be a continuous functional that attains its minimum for every finite set  $Y \subset \mathbb{R}^2$  for the Delaunay triangulation of  $Y$ . Then the density  $\bar{F}$  on  $\Theta(X)$ , where  $X \subset \mathbb{R}^2$  is a Delaunay set, attains its minimum for the Delaunay triangulation of  $X$ .*

*Proof.* Let  $T \in \Theta(X)$  be a triangulation with parameter  $q$ , and consider the simplicial complex  $T_\alpha \subset T$  that consists of all triangles, edges, and vertices of  $T$  contained in  $B_\alpha$ . We consider the convex hull of the vertices of  $T_\alpha$ , which we denote as  $C_\alpha$ . The difference between  $C_\alpha$  and the union of triangles in  $T_\alpha$  consists of polygons, and since any polygon can be triangulated without adding vertices,  $T_\alpha$  can be extended to a triangulation  $T'_\alpha$  of the same set of vertices. Write  $K_\alpha$  for the number of triangles in  $T_\alpha$ . Since the circumradius of each triangle is bounded from above, and the lengths of its edges are bounded from below, the area of each triangle is at least some constant. It follows that  $K_\alpha$  is at most some constant times  $\alpha^2$ . The circle bounding  $B_\alpha$  intersects at most some constant times  $\sqrt{K_\alpha}$  of the triangles in  $T$ , which implies that  $T'_\alpha$  has at most some constant times  $\sqrt{K_\alpha}$  triangles in addition to those in  $T_\alpha$ . Using the continuity of the functional, it follows that the limit of the ratio  $F(T_\alpha)$  over  $F(T'_\alpha)$ , for  $\alpha$  going to infinity, is equal to 1. By the assumption,  $F(T'_\alpha)$  is no less than the value of  $F$  on the Delaunay triangulation of the same set of vertices, which completes the proof.  $\square$

We remark that there are non-convex polytopes in dimension  $d > 2$  that cannot be triangulated without adding new vertices. They constitute the main difficulty in extending the Theorem to general dimension. The theorem and the results stated in the papers [1, 2, 3, 4, 5] yield

**Corollary.** *Let  $\Delta$  be a triangle with barycenter  $b$ , circumcenter  $c$ , and edges of lengths  $a_1, a_2, a_3$ . Let us consider the following functionals:*

- (1)  $F_1(\Delta) = \mathcal{R}^a(\Delta)$ , where  $\mathcal{R}(\Delta)$  is the circumradius and  $a > 0$ ;
- (2)  $F_2(\Delta) = (a_1^2 + a_2^2 + a_3^2)/\text{area}(\Delta)$ , where  $\text{area}(\Delta)$  is the area of  $\Delta$ ;
- (3)  $F_3(\Delta) = -\rho(\Delta)$ , where  $\rho(\Delta)$  is the inradius of  $\Delta$ ;
- (4)  $F_4(\Delta) = (a_1^2 + a_2^2 + a_3^2) \text{area}(\Delta)$ ;

- (5)  $F_5(\Delta) = \mathcal{R}(\Delta)^a \text{area}(\Delta)$ , where  $a \geq 1$ ;  
(6)  $F_6(\Delta) = \|b(\Delta) - c(\Delta)\|^2 \text{area}(\Delta)$ . Then the densities  $\bar{F}_i$ , for  $i = 1, 2, \dots, 6$ , achieve their minima on the Delaunay triangulations of Delaunay sets in the plane.

For finite sets, the optimality of the functionals  $F_1$  and  $F_2$  has been shown in [3], of  $F_3$  it has been shown in [2], of  $F_4$  it has been shown [5], and of  $F_5$  and  $F_6$  it has been shown in [4].

## References

- [1] Delone, B.N., *UMN* **3** (1937), 16-62.
- [2] Lambert, T., The Delaunay triangulation maximizes the mean inradius, In "Proc. 6th Canad. Conf. Comput. Geom., 1994", 201-206.
- [3] Musin, O.R., Properties of the Delaunay triangulation, In "Proc. 13th Ann. ACM Sympos. Comput. Geom., 1997", 424-426.
- [4] Musin, O.R., About optimality of Delaunay triangulations, in *Geometry, Topology, Algebra and Number Theory Applications*. Internat. Conf. dedicated to the 120th anniversary of B.N. Delone, 2010, 166-167. <http://delone120.mi.ras.ru/app/Musin.pdf>
- [5] Rajan, V.T., Optimality of the Delaunay triangulation in  $\mathbb{R}^d$ , *Discrete Comput. Geometry* **12** (1994), 189-202.